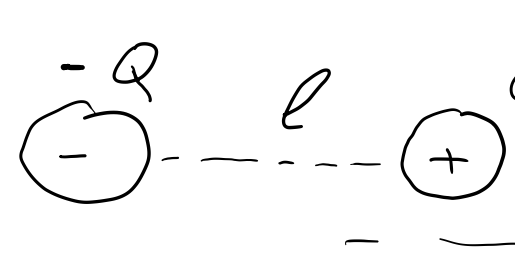
Emission of electromagnetic wave

So far we have been studying propagation of light or electro-magnetic wave in empty space, i.e. vacuum. That means we used Maxwell's equations where $\rho=0$ or $j=0$

Now we will try to understand how do electro-magnetic waves appear.

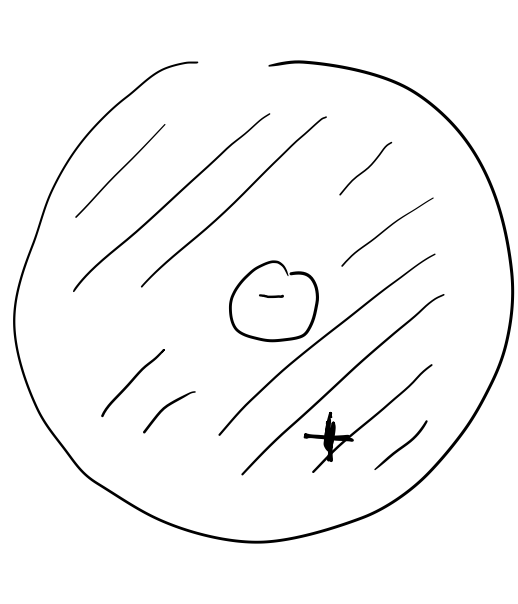
Obviously, we need to take the simplest model – dipole oscillator:



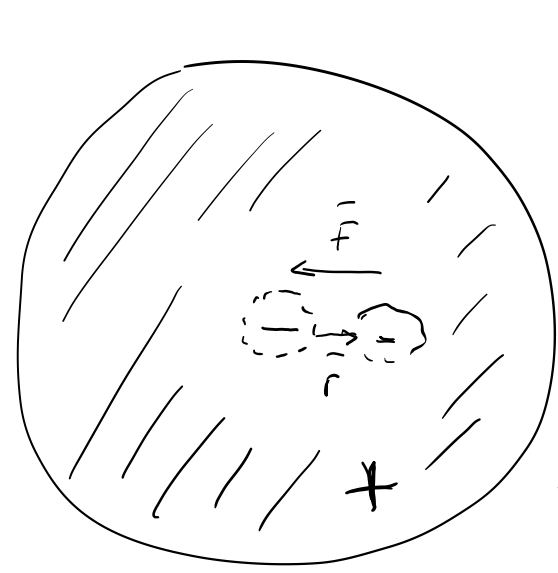
p - dipole model of such oscillator

$$\boxed{p = Ql}$$

Now, how can we envision atom as dipole oscillator? The simplest „artificial" solution has been proposed by Thompson and is called Thompson atom model.



According to Thompson, atom is positive cloud with negative electron in a center



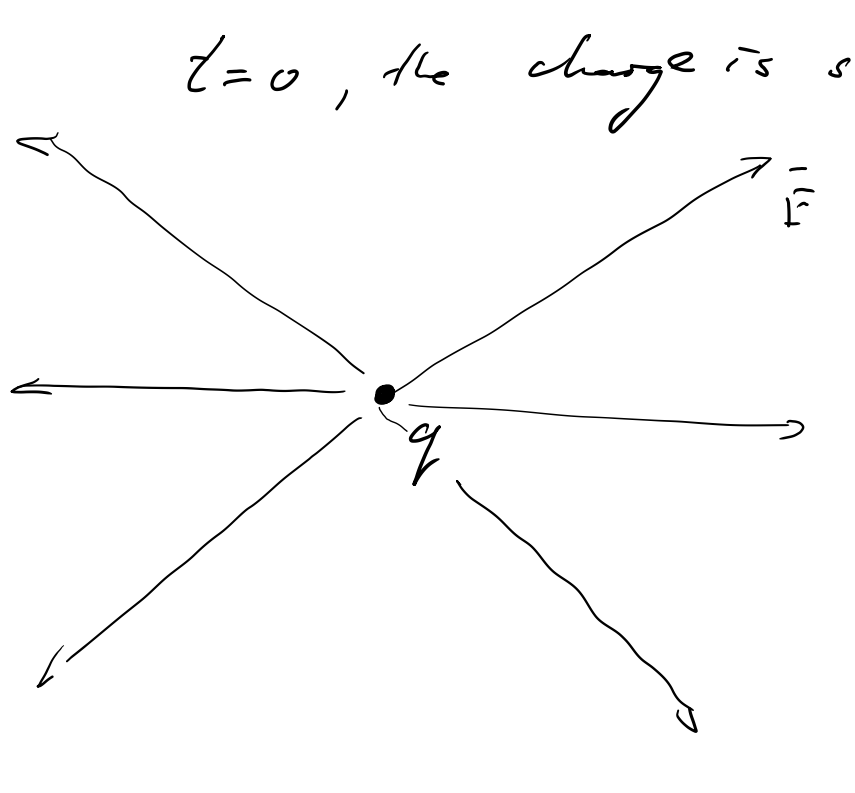
If we apply some force to displace electron from the center, then another force appear that returns the electron back to its equilibrium position.

$$\bar{F} = -k\bar{r}$$

k is restoring coefficient.

Let's see how we can visualize emission of e/m wave using Thompson model:

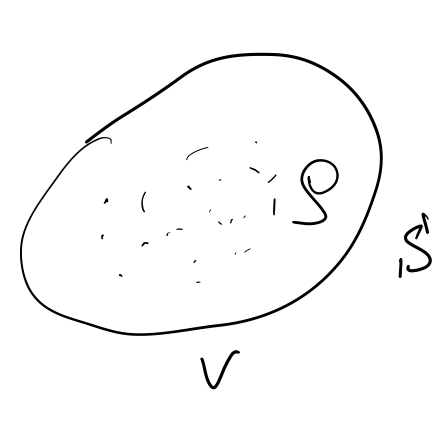
$t=0$, the charge is still



Its electric field will be represented by field lines $\bar{E}$

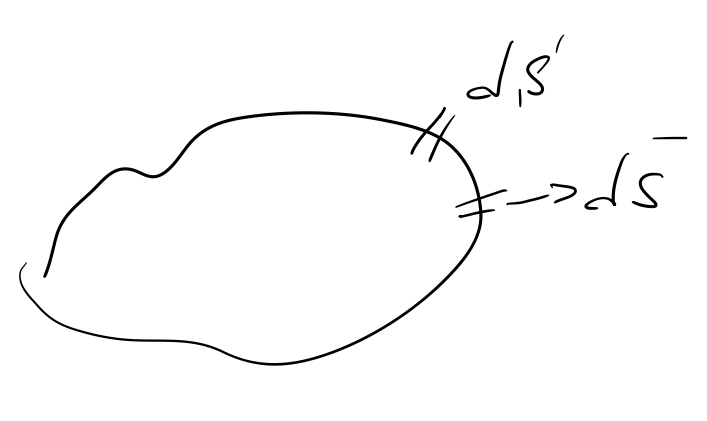
How $\bar{E}$ will change with $\bar{r}$?

$$\operatorname{div}\bar{E} = \frac{\rho}{\varepsilon_0}$$



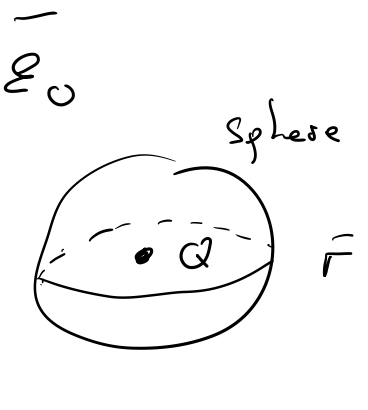
$$\int_V \operatorname{div}\bar{E}\,dV = \frac{1}{\varepsilon_0}\int_V \rho\, dV$$ Ostrogradsky Gauss

$$\oint_S \bar{E}\,d\bar{S} = \frac{1}{\varepsilon_0}\int \rho\, dV$$



The flux of electric field through closed surface S is determined by the total charge within V.

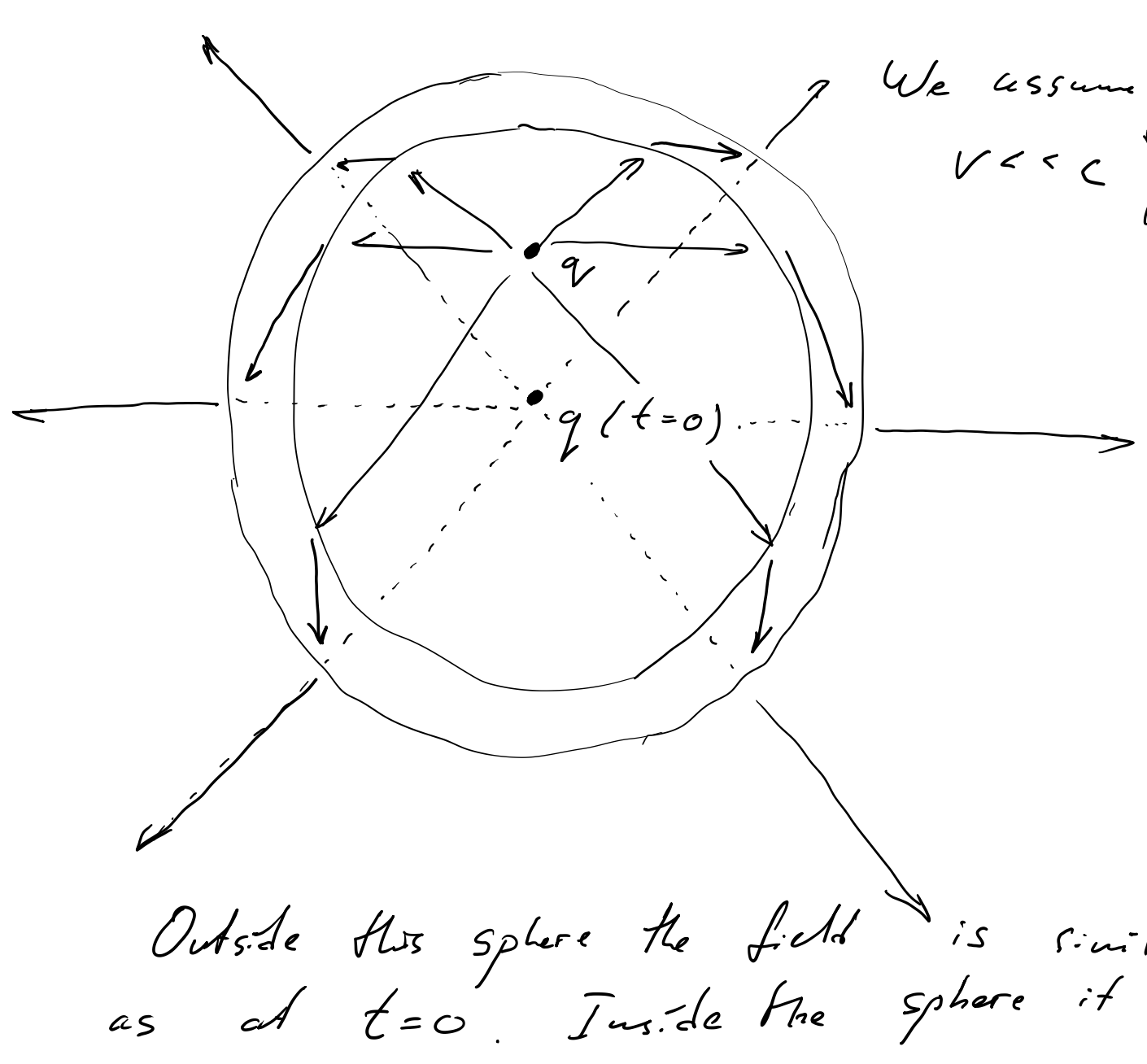
$$\oint_S \bar{E}\,d\bar{S} = \frac{1}{\varepsilon_0}\int \rho\, dV = \frac{Q}{\varepsilon_0}$$



$$4\pi r^2(\bar{E}\bar{r}) = \frac{Q}{\varepsilon_0}$$

$$\bar{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\bar{r} \rightarrow \text{Coulomb}$$

At certain moment the charge is moved by force $\bar{F}$ for the time τ and then left going with certain speed $v = const$ as force is not applied anymore.



We assume that $v << c$ ∇ 0

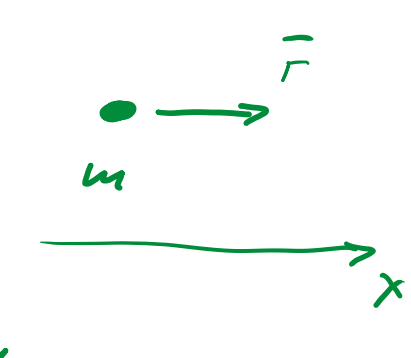
Outside this sphere the field is similar as at $t=0$. Inside the sphere it is also static as $v = const$.

It is as field $\bar{E}$ at $v\tau$.

The breaks of force lines that travel away from the charge are actually the e/m wave. Now, we know, that if we apply restoring force similar to $\bar{F} = -k\bar{r}$, then the oscillator will have harmonic oscillations.

Little mathematical walkaround

$\bar{F} = -k\bar{r}$



Then from the second Newton's law:

$$-kx = ma$$

$$m\frac{\partial^2 x}{\partial t^2} + kx = 0 \Rightarrow \frac{\partial^2 x}{\partial t^2} + \omega^2 x = 0$$

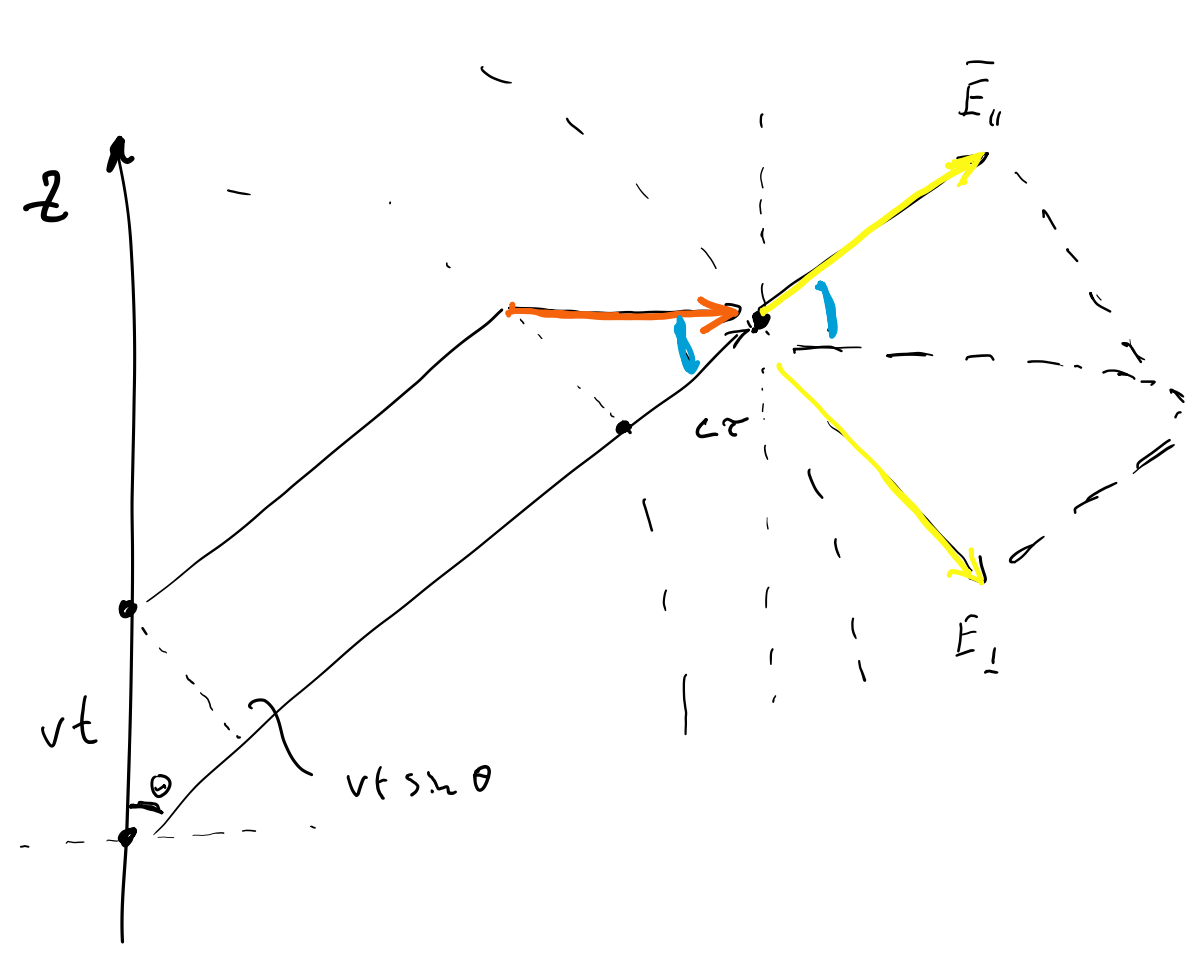
Solution $\rightarrow x(t) = A\cos(\omega t + \varphi)$

Let's have a look at one of the lines that have angle θ with respect to the charge movement.

τ - time of charge brief acceleration

t - time charge moves with $v = const$ after

$$t >> \tau$$



As it is clear

$$\frac{E_{\perp}}{E_{\parallel}} = \frac{vt\sin\theta}{c\tau}$$ (Explain!)

$$\begin{cases} E_{\parallel} = \dfrac{q}{4\pi\varepsilon_0 r^2} \\ r = ct \end{cases} \Rightarrow E_{\perp} = \frac{qv}{4\pi\varepsilon_0 c^3 t\tau}\sin\theta$$

1) Since $\frac{v}{\tau} = a$, acceleration of moving charge, when the force was applied.

2) $t = \frac{r}{c}$

$$\Downarrow$$

$$\begin{cases} E_{\perp} = \dfrac{1}{4\pi\varepsilon_0}\cdot\dfrac{qa}{c^2 r}\sin\theta \\ E_{\parallel} = \dfrac{q}{4\pi\varepsilon_0 r^2} \end{cases}$$

What does it mean?

1) $E_{\perp}$ is dependent on a. However a is the acceleration that happened at $t=0$. The wave emitted at $t=0$ will reach the observation point at $\frac{r}{c}$.

$$a(t') = a(t - r/c)$$

We can harmonically oscillate the charge:

$$z = z_0\cos\omega t$$

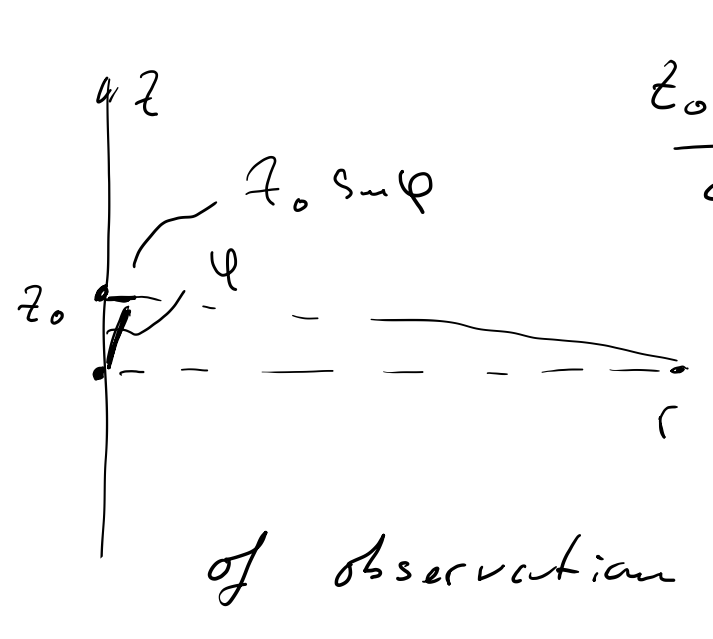
$$a(t) = \frac{\partial^2 z}{\partial t^2} = -\omega^2 z_0\cos\omega t$$

$$E_{\perp} = \frac{1}{4\pi\varepsilon_0 c^2 r}Q\omega^2 z_0\cos\left(\omega\left(t - \frac{r}{c}\right)\right)\sin\theta = A\cos(\omega t - kr)$$

Surface of equal phase $\omega\left(t - \frac{r}{c}\right) = \omega t - kr = const$

Spheres with the center at the center of coordinates, expanding with $c = \frac{\omega}{k}$

If we have charge q that oscillates with amplitude z_0, then it is important that $z_0 << ?$.



$$\frac{z_0\sin\varphi}{c} << T \Rightarrow z_0\sin\varphi << \lambda$$

In this case the delay time of wave in point of observation is r/c. In other words it is important $v << c$.

This charge is called dipole oscillator.

E/M field emitted only by accelerating charge can

$$\begin{cases} v << c \\ z_0 << \lambda \end{cases}$$ dipole approximation

2) $E_{\perp} \sim \frac{1}{r}$

$$E_{\perp} = \frac{1}{4\pi\varepsilon_0}\frac{qa}{c^2 r}\sin\theta = \underbrace{\frac{1}{4\pi\varepsilon_0}\frac{Qz_0\omega^2}{c^2 z}}_{A \sim \frac{1}{r}}\cos(\omega t - kz)$$

Moreover

$$E_{\parallel} \sim \frac{1}{r^2} \rightarrow 0$$

$$\bar{E}_{\perp} \sim \frac{1}{r}$$

It means it is spherical wave

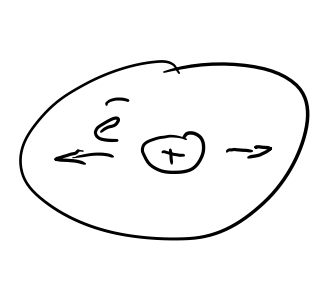
We can add another charge $-q$ to the center of coordinates. However, since it is static, it will only influence $\bar{E}_{\parallel}$, that is decaying as $\frac{1}{r^2}$, while $E_{\perp} \sim \frac{1}{r}$

From this point it is not important if we move charge q or $-q$, or, actually change the charge itself as $q = q_0\cos\omega t$ while keeping its position fixed.

Changing $q \rightarrow$ radio antenna
$q = q_0\cos\omega t$

Changing $z_0 \rightarrow$ atom emission

$z = z_0\cos\omega t$

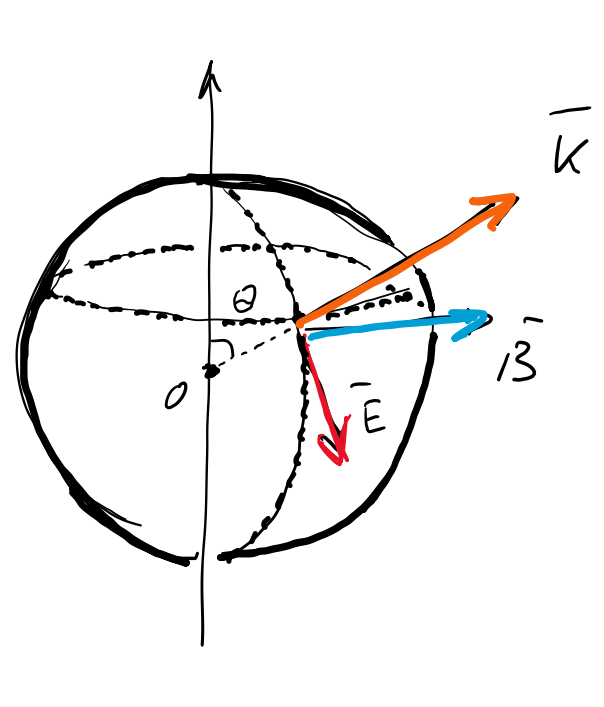


3) $E_{\perp} \sim \sin\theta$

It means the oscillator emits more in direction perpendicular to the movement.

When $r >> \lambda$ spherical wave can be seen as plane:

In polar coordinates



$$\begin{cases} E_r = 0 \\ E_\theta = \dfrac{1}{4\pi\varepsilon_0}\cdot\dfrac{\ddot{P}}{c^2 r}\sin\theta \\ E_\varphi = 0 \end{cases} \qquad \begin{cases} B_r = 0 \\ B_\theta = 0 \\ B_\varphi = \dfrac{1}{4\pi\varepsilon_0}\cdot\dfrac{\ddot{P}}{c^2 r}\sin\theta \end{cases}$$

$$\bar{B} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{c^2 r}[\ddot{\bar{P}}\,\bar{r}_0]$$

$$\bar{r}_0 = \frac{\bar{r}}{r}$$

Thus $\boxed{\bar{E} = c[\bar{B}\,\bar{r}_0]}$

It is important to remember, that although field of emission of dipole oscillator is sphere, it does not possess spherical symmetry. Modulus of $\bar{E}$ and $\bar{B}$ are different at different positions of sphere and depend on θ.

The energy flux will be:

$$I = \langle S\rangle = \varepsilon_0 c\langle[\bar{E}\bar{B}]\rangle =$$

$$= \varepsilon_0 c^2\langle EB\rangle = \varepsilon_0 c\langle E^2\rangle = \varepsilon_0 c\frac{\langle\ddot{p}^2\rangle\sin^2\theta}{(4\pi)^2\varepsilon_0^2 c^4 r^2}$$

Since $p = p_0\cos(\omega t)$

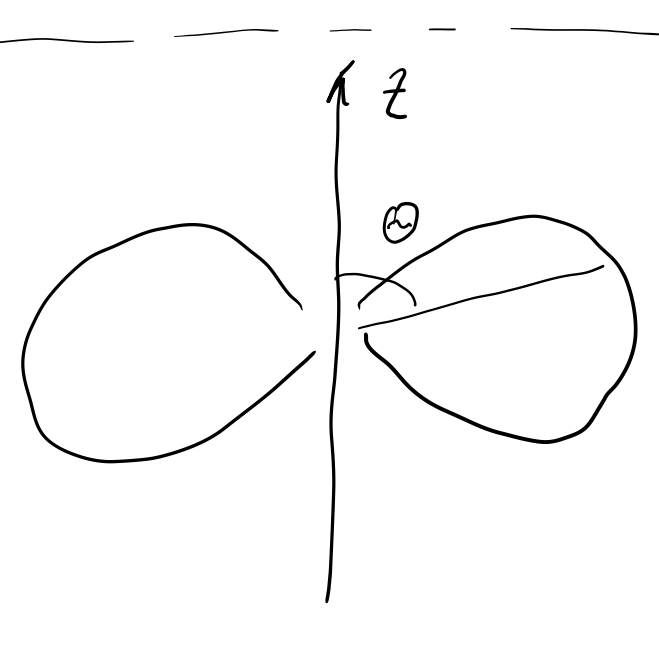
$$p(t') = p_0\cos\left(\omega\left(t - \frac{r}{c}\right)\right) = p_0\cos(\omega t - kr)$$

$$\ddot{p} = \omega^2 p_0\cos(\omega t - kr)$$

$$\Downarrow$$

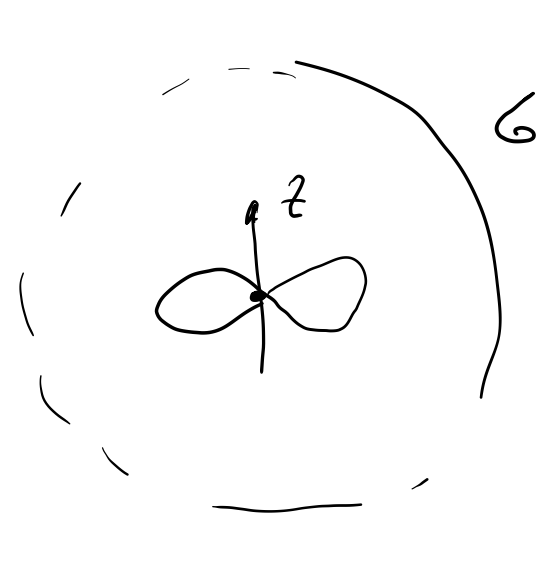
$$\langle\ddot{p}^2\rangle = \frac{1}{2}\omega^4 p_0^2$$

$$\boxed{I = \frac{\omega^4 p_0^2}{32\pi^2\varepsilon_0 c^3 r^2}\sin^2\theta}$$



Now, let's determine the whole energy of emission

$$P = \int_\sigma I\,d\sigma =$$



$$= \int_0^{2\pi}\int_0^{\pi} I r^2\sin\theta\,d\theta\,d\varphi =$$

integrating on θ

integrating on φ

$$= \frac{\omega^4 p_0^2}{32\pi^2\varepsilon_0 c^3}\,2\pi\underbrace{\int_0^{\pi}\sin^3\theta\,d\theta}_{\frac{4}{3}} = \frac{\omega^4 p_0^2}{12\pi\varepsilon_0 c^3}$$

What does it mean?

① $P \sim p_0^2$

$P \sim \omega^4 \sim \frac{1}{\lambda^4}$

This is very important when we will look into scattering of light.

That is why sky is blue during day and red on the sunset.

② P does not depend on r.

This means that energy flux through any sphere R is the same. If there is no conductors or charges in the space, there is no way to lose energy or transfer it to another form.